

Complex Number (Problem Set)

Srijan Chattopadhyay

August 19, 2026

1. Suppose $A(z_1), B(z_2), C(z_3), D(z_4)$ are 4 points in the argand plane such that AB is perpendicular to CD . Prove that $\frac{z_1 - z_2}{z_3 - z_4}$ is purely imaginary.
2. Show that the point a' is the reflection of the point a in the line $z\bar{b} + \bar{z}b + c = 0$, if $a'\bar{b} + \bar{a}'b + c = 0$.
3. If $|z - 2 + i| \leq 2$, find the greatest and least value of $|z|$. (Can you do it without algebra?)
4. If ω is complex cube root of unity and a, b, c are such that

$$\frac{1}{a + \omega} + \frac{1}{b + \omega} + \frac{1}{c + \omega} = 2\omega^2$$

and

$$\frac{1}{a + \omega^2} + \frac{1}{b + \omega^2} + \frac{1}{c + \omega^2} = 2\omega$$

then find the value of

$$\frac{1}{a + 1} + \frac{1}{b + 1} + \frac{1}{c + 1}.$$

5. If a, b, c are distinct integers and $\omega (\neq 1)$ is a cube root of unity, find minimum value of

$$|a + b\omega + c\omega^2| + |a + b\omega^2 + c\omega|$$

6. If A, B and C represent the complex numbers z_1, z_2 and z_3 respectively on the complex plane and the angles at B and C are each equal to $\frac{1}{2}(\pi - \alpha)$, then prove that

$$(z_2 - z_3)^2 = 4(z_3 - z_1)(z_1 - z_2) \sin^2\left(\frac{\alpha}{2}\right).$$

7. If z_1 and z_2 are two complex numbers such that

$$\left| \frac{z_1 - z_2}{z_1 + z_2} \right| = 1,$$

then prove that

$$\frac{iz_1}{z_2} = k, \text{ where } k \text{ is a real number.}$$

Find the angle between the lines from the origin to the points $z_1 + z_2$ and $z_1 - z_2$ in terms of k .

8. If a is a complex number such that $|a| = 1$, then find the value of a , so that the equation $az^2 + z + 1 = 0$ has one purely imaginary root.
9. If $n \in \mathbb{N} > 1$, find the sum of real parts of the roots of the equation $z^n = (z + 1)^n$.
10. Interpret the following equations geometrically on the Argand plane.
 - (i) $|z - 1| + |z + 1| = 4$
 - (ii) $\arg(z + i) - \arg(z - i) = \frac{\pi}{2}$
 - (iii) $1 < |z - 2 - 3i| < 4$

- (iv) $\frac{\pi}{4} < \arg(z) < \frac{\pi}{3}$
 (v) $\log_{\cos \frac{\pi}{3}} \left\{ \frac{|z-1|+4}{3|z-1|-2} \right\} > 1$

11. If ω is the n th root of unity and z_1, z_2 are any two complex numbers, then prove that

$$\sum_{k=0}^{n-1} |z_1 + \omega^k z_2|^2 = n \{ |z_1|^2 + |z_2|^2 \}, \quad \text{where } n \in \mathbb{N}.$$

12. Let $\sum_{i=1}^4 a_i = 0$ and $\sum_{j=1}^4 a_j z_j = 0$, then prove that z_1, z_2, z_3 and z_4 are concyclic, if

$$a_1 a_2 |z_1 - z_2|^2 = a_3 a_4 |z_3 - z_4|^2.$$

13. Prove that all roots of $11z^{10} + 10iz^9 + 10iz - 11 = 0$ have unit modulus (or equivalent $|z| = 1$).

14. Let a, b, c be real numbers. Let z_1, z_2, z_3 be complex numbers such that $|z_k| = 1$ ($k = 1, 2, 3$) and $\frac{z_1}{z_2} + \frac{z_2}{z_3} + \frac{z_3}{z_1} = 1$. Find $|az_1 + bz_2 + cz_3|$.

15. Let complex numbers α and $\frac{1}{\alpha}$ lie on the circles $(x-x_0)^2 + (y-y_0)^2 = r^2$ and $(x-x_0)^2 + (y-y_0)^2 = 4r^2$ respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| = ?$

16. If $|z| \geq 3$, then determine the least value of $|z + \frac{1}{z}|$.

17. If $|z + \frac{4}{z}| = 2$, then find the minimum and maximum values of $|z|$.

18. For any integer k , let $\alpha_k = \cos(\frac{k\pi}{7}) + i \sin(\frac{k\pi}{7})$. Then find the value of $\frac{\sum_{i=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^3 |\alpha_{4k-1} - \alpha_{4k-2}|}$.

19. Let $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Suppose $S = \{z \in \mathbb{C} : z = \frac{1}{a+ibt}, t \in \mathbb{R}, t \neq 0\}$. If $z = x + iy$ and $z \in S$, find locus of z .

20. Let z be a complex number satisfying $|z|^3 + 2z^2 + 4\bar{z} - 8 = 0$, where $\bar{z}$ denotes the complex conjugate of z . Let the imaginary part of z be nonzero. $|z|^2 = ?$ $|z - \bar{z}|^2 = ?$ $|z|^2 + |z + \bar{z}|^2 = ?$ $|z + 1|^2 = ?$

21. $z_1, z_2, \dots$ be a sequence of complex numbers defined by $z_1 = i$ and $z_{n+1} = z_n^2 + i$ for $n \geq 1$. Then $|z_{2024} - z_1| = ?$

22. Consider the two subsets of $\mathbb{C}$, $A = \{\frac{1}{z} : |z| = 2\}$ and $B = \{\frac{1}{z} : |z - 1| = 2\}$. Identify A and B as geometric figures.

23. Find the minimum possible value of $|z|^2 + |z - 3|^2 + |z - 6i|^2$, where z is a complex number.

24. Let $0 < a_1 < a_2 < \dots < a_n$ be real numbers. Show that the equation $\frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \dots + \frac{a_n}{a_n - x} = 2015$ has exactly n real roots.

25. If z_1, z_2, z_3 are non-zero complex numbers such that

$$\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}.$$

Then prove that z_1, z_2, z_3 lie on a circle passing through the origin.

26. What is the area of the figure in the complex plane enclosed by the origin and the set of all points $\frac{1}{z}$ such that

$$(1 - 2i)z + (-2i - 1)\bar{z} = 6i?$$

27. What is the area of the region in the complex plane consisting of all points z satisfying both

$$\left| \frac{1}{z} - 1 \right| < 1 \quad \text{and} \quad |z - 1| < 1?$$

($|z|$ denotes the magnitude of a complex number, i.e. $|a + bi| = \sqrt{a^2 + b^2}$.)

28. Let $z = x + iy$ be a complex number with x and y rational and with $|z| = 1$.

(a) Find two such complex numbers. (b) Show that $|z^{2n} - 1| = 2|\sin n\theta|$, where $z = e^{i\theta}$.